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The massless spectrum of covariant superstrings

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Abstract: We obtain the correct cohomology at any ghost number for the open and closed covariant superstring, quantized by an approach which we recently developed. We define physical states by the usual condition of BRST invariance and a new condition involving a new current which is related to a grading of the underlying affine Lie algebra.

Keywords: Superstrings and Heterotic Strings, Conformal Field Models in String Theory, Space-Time Symmetries, BRST Symmetry.

Contents

1. Introduction

Recently, we developed a new approach to the long-standing problem of the covariant quantization of the superstring [1]. The formulation of Berkovits of super-Poincaré covariant superstrings in $9+1$ dimensions [2] is based on a *free* conformal field theory on the world-sheet and a nilpotent BRST charge which defines the physical vertices as representatives of its cohomology. In addition to the conventional variables x^m and θ^α of the Green-Schwarz formalism, a conjugate momentum p_α for θ^α and a set of commuting ghost fields λ^α are introduced. The latter are complex Weyl spinors satisfying the pure spinor conditions $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$ (cf. for example [3]). This equation can be solved by decomposing λ with respect to a non-compact U(5) subgroup of SO(9,1) into a singlet $\underline{1}$, a vector $\underline{5}$, and a tensor $\underline{10}$. The vector can be expressed in terms of the singlet and tensor, hence there are 11 independent complex variables in λ^α.

Since the presence of the non-linear constraint $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$ makes the theory unsuitable for a path integral quantization and higher loop computations, we relaxed the pure spinor condition by adding further ghosts. We were naturally led to a finite set of extra fields, but the BRST charge Q of this system was not nilpotent, and the central charge of the conformal field theory did not vanish. The latter problem was solved by adding one more extra ghost system, which we denoted by η^m and ω^m_z. The former problem was solved by introducing yet another new ghost pair, b and c_z, which we tentatively associated with the central charge generator in the affine superalgebra which plays an essential role in the superstring [4].

The BRST charge is linear in c_z, and without further conditions on physical states the theory would be trivial. We proposed that physical states belong not only to the BRST cohomology ($Q\,|\psi\rangle = 0$, but $|\psi\rangle \neq Q\,|\phi\rangle$), but also that the deformed stress tensor $T + \mathcal{V}^{(0)}$, where $\mathcal{V}^{(0)}$ is a vertex operator, satisfies the usual OPE of a conformal spin 2 tensor. (The latter condition is weaker than the requirement that vertex operators be primary fields with conformal spin equal to 1).

In this letter we propose another definition of physical states. We retain the BRST condition, but we replace the stress tensor condition by the requirement that the physical states belong to a subspace $\mathcal{H}'$ of the entire linear space $\mathcal{H}$ of vertex operator. The latter can be decomposed w.r.t. a grading naturally associated with the underlying affine algebra (cf. [5]) as $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$, with negative and non-negative grading, respectively. The BRST charge $Q = \sum_{n\geq 0} Q_n$ contains only terms Q_n with non-negative grading, hence one can consistently consider the action of Q in $\mathcal{H}_+$. The physical space is identified with the cohomology group $H(Q, \mathcal{H}_+)$, namely

$$\begin{aligned} Q|\psi\rangle &= 0\,, \qquad |\psi\rangle \in \mathcal{H}_+\,, \\ |\psi\rangle &\neq Q\,|\phi\rangle\,, \qquad |\phi\rangle \in \mathcal{H}_+\,. \end{aligned} \tag{1.1}$$

We show that with these conditions we obtain in a very simple way the correct massless spectrum of the open and closed superstring. In particular, our results agree with the work of Berkovits and collaborators [2], who obtains the correct field equations for the massless states $\gamma^{\alpha\beta}_{mnrpq} D_\alpha A_\beta = 0$, but defines physical states by only the BRST condition on the unintegrated vertex $\mathcal{U}^{(1)} = \lambda^\alpha A_\alpha(x,\theta)$ in terms of pure spinors λ^α. In their work the superfields A_m and W^α (the latter is denoted by A^α in [1]) of the open string are expressed in term of A_α by the relations of $d = 10$ superspace,

$$A_m = \frac{1}{8}\gamma^{\alpha\beta}_m D_\alpha A_\beta\,, \qquad W^\alpha \equiv A^\alpha = \frac{1}{10}\gamma^{m,\alpha\beta}\left(D_\beta A_m - \partial_m A_\beta\right), \tag{1.2}$$

whereas in our work these same relations follow from the physical state conditions. We obtain corresponding results for closed (type-II) strings, which also agree with [2] and [6].

Then we consider the sectors with ghost number different from one, and obtain in all these sectors the correct results.

We also give a derivation of the η^m, ω^m_z terms in the action, based on a local symmetry which has been found in the constrained spinor approach [7]. This derivation leads us to replace the pair (η^m, ω^m_z) in [1] with (η^m_z, ω^m).

2. The new current and grading

We based our approach on the following affine superalgebra [4]

$$\begin{aligned} d_\alpha(z) d_\beta(w) &\sim -\frac{\gamma^m_{\alpha\beta}\Pi_m(w)}{z-w}\,, \qquad d_\alpha(z)\Pi^m(w) \sim \frac{\gamma^m_{\alpha\beta}\partial\theta^\beta(w)}{z-w}\,, \\ \Pi^m(z)\Pi^n(w) &\sim -\frac{1}{(z-w)^2}\eta^{mn}k\,, \qquad d_\alpha(z)\partial_w\theta^\beta(w) \sim \frac{1}{(z-w)^2}\delta_\alpha{}^\beta k\,, \\ \Pi^m(z)\partial_w\theta^\beta(w) &\sim 0\,, \qquad \partial_z\theta^\alpha(z)\partial_w\theta^\beta(w) \sim 0\,, \end{aligned} \tag{2.1}$$

where $\sim$ denotes the singular contributions to the OPE's. This algebra has a natural grading defined as follows: $d_\alpha(z)$ has grading $1/2$, $\Pi^m(z)$ has grading 1, $\partial_z\theta^\alpha(z)$ has grading $3/2$, and the central charge k (which numerically is equal to unity) has grading 2.

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The corresponding ghost systems are $(\lambda^\alpha, \beta_{z\alpha})$, (ξ^m, β_{zm}), $(\chi_\alpha, \kappa_z^\alpha)$, and (c_z, b). We thus define the following grading for the ghosts and corresponding antighosts

$$
\begin{aligned}
&\mathrm{gr}(\lambda^\alpha) = \frac{1}{2}\,, \qquad \mathrm{gr}(\xi^m) = 1\,, \qquad \mathrm{gr}(\chi_\alpha) = \frac{3}{2}\,, \qquad \mathrm{gr}(c_z) = 2\,,\\
&\mathrm{gr}(\beta_\alpha) = -\frac{1}{2}\,, \qquad \mathrm{gr}(\beta_m) = -1\,, \quad \mathrm{gr}(\kappa^\alpha) = -\frac{3}{2}\,, \qquad \mathrm{gr}(b) = -2\,.
\end{aligned} \tag{2.2}
$$

We also need the ghost ω^m and the antighost η_z^m, although this pair does not seem to correspond to a generator. We assign the grading $\mathrm{gr}(\eta_z^m) = -2$ and $\mathrm{gr}(\omega^m) = 2$ for the following reason. In [1], we relaxed the pure spinor constraint by successively adding quartets starting from $(\lambda_+, \lambda_{[ab]}; \beta^+, \beta^{[ab]})$ of [2] (the indices a, b belong to the fundamental representation of the U(5) subgroup of SO(1,9)), and adding the fields $(\lambda^a, \beta_a; \xi^a, \beta'_a)$ with grading $(1/2, -1/2, 1, -1)$. This procedure yields the covariant spinors λ^α and β_α, but now the fields (ξ^a, β'_a) are non-covariant w.r.t. SO(9,1). Thus, we added the quartet $(\xi_a, \beta'^a; \chi_a, \kappa^a)$ with grading $(1, -1, 3/2, -3/2)$. The spinors (χ_a, κ^a) are part of a covariant spinor and the missing parts are introduced by adding the quartets (χ^+, κ_+, c, b) and $(\chi^{[ab]}, \kappa_{[ab]}, \eta^m, \omega^m)$, both with grading $(3/2, -3/2, 2, -2)$. In this way, we obtain the covariant fields $\lambda^\alpha = (\lambda_+, \lambda^a, \lambda_{[ab]})$; $\beta_\alpha = (\beta^+, \beta_a, \beta^{[ab]})$; $\xi^m = (\xi^a, \xi_a)$; $\beta^m = (\beta'^a, \beta'_a)$; $\chi_\alpha = (\chi^+, \chi_a, \chi^{[ab]})$; $\kappa^\alpha = (\kappa_+, \kappa^a, \kappa_{ab})$; b, c and η_m, ω^m.

As usual for a conformal field theory, it is natural to introduce a current whose OPE's with the ghost and antighosts reproduce the grading assignments in (2.2)

$$
j_z^{\mathrm{grad}} = -\frac{1}{2}\beta_{z,\alpha}\lambda^\alpha - \beta_{z,m}\xi^m - \frac{3}{2}\kappa_z^\alpha\chi_\alpha - 2bc_z - 2\eta_z^m\omega_m\,. \tag{2.3}
$$

Independent confirmation that this current might be important comes from the cancellation of the anomaly (namely the terms with $(z-w)^{-3}$) in the OPE of the stress energy tensor $T_{zz}(z)$ (cf. ref. [1, eqs. (1-3)]) with j_z^{grad}. In fact, one finds

$$
c^{\mathrm{grad}} = \frac{1}{2}\times(+16)_{\lambda\beta} + 1\times(-10)_{\xi\beta} + \frac{3}{2}\times(+16)_{\kappa\chi} + 2\times(-1)_{bc} + 2\times(-10)_{\eta\omega} = 0\,. \tag{2.4}
$$

The requirement that the vertex operators contain only terms with non-negative grading will lead to the correct massless spectrum. It will also severely restrict the contribution of the vertex operators to correlation functions (in the usual RNS approach ghost insertions are needed to compensate the anomaly in the ghost current, whereas here we anticipate to need insertions of fields in $\mathcal{H}_-$ to compensate the non-negative grading of vertex operators $\mathcal{U}^{(1)} \in \mathcal{H}_+$).

All the terms in the stress tensor $T_{zz}(z)$ and in the ghost current

$$
\begin{aligned}
T_{zz} &= -\frac{1}{2}\Pi_z^m\Pi_{mz} - d_{z\alpha}\partial_z\theta^\alpha - \beta_{zm}\partial_z\xi^m - \beta_{z\alpha}\partial_z\lambda^\alpha - \kappa_z^\alpha\partial_z\chi_\alpha + \partial_z b\, c_z - \eta_z^m\partial_z\omega_m\,,\\
J_z^{gh} &= -\left(\beta_{mz}\xi^m + \kappa_z^\alpha\chi_\alpha + \beta_{z\alpha}\lambda^\alpha + b\,c_z + \eta_z^m\omega_m\right),
\end{aligned} \tag{2.5}
$$

have grading zero, since they are sums of terms of ghost and antighost pairs with opposite grading. On the other hand, the terms in the current $j_z^B(z)$ (cf. [1, eq. (1.2)]) and the field

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$B_{zz}(z)$ have different grading.[1] For instance, the BRST current can be decomposed in the following pieces $j^B_z(z) = \sum_{n=0}^2 j_z^{B,(n)}(z)$

$$
\begin{aligned}
j_z^{B,(0)}(z) &= -\xi^m \kappa_z^\alpha \gamma_{m\alpha\beta}\lambda^\beta - \frac{1}{2}\lambda^\alpha \gamma^m_{\alpha\beta}\lambda^\beta \beta_{zm} - \\
&\quad - \frac{1}{2} b \left(\xi^m \partial_z \xi_m - \frac{3}{2}\chi_\alpha \partial_z \lambda^\alpha + \frac{1}{2}\partial_z \chi_\alpha \lambda^\alpha \right) - \frac{1}{2}\partial_z \left(b\, \chi_\alpha \lambda^\alpha \right), \\
j_z^{B,(1/2)}(z) &= \lambda^\alpha d_{z\alpha}, \qquad j_z^{B,(1)}(z) = -\xi^m \Pi_{zm}, \\
j_z^{B,(3/2)}(z) &= -\chi_\alpha \partial_z \theta^\alpha, \qquad j_z^{B,(2)}(z) = c_z.
\end{aligned} \tag{2.6}
$$

It is clear that all terms in $j^B_z(z)$ have non-negative grading.

3. Massless spectrum of the open superstring

We now turn to the determination of the massless cohomology for the open superstring, namely we will compute $H(Q, \mathcal{H}_+)$ for any ghost number.

Following the RNS approach, the cohomology $H^{(1)}(Q, \mathcal{H}_+)$ at ghost number 1 should be identified with the physical fields of the open superstring. In particular, the massless spectrum coincides with spin zero world-sheet fields $\mathcal{U}^{(1)}(z)$. The most general scalar expression in $\mathcal{H}_+$ is

$$
\begin{aligned}
\mathcal{U}^{(1)}(z) &= \lambda^\alpha A_\alpha + \xi^m A_m + \chi_\alpha W^\alpha + \omega^m B_m + \\
&\quad + b \left(\xi^m \xi^n F_{mn} + \lambda^\alpha \chi_\beta F_\alpha^{\ \beta} + \chi_\alpha \xi^m F_m^{\ \alpha} + \chi_\alpha \chi_\beta F^{\alpha\beta} \right) + \\
&\quad + b\, \omega^m \left(\lambda^\alpha G_{m\alpha} + \xi^n G_{mn} + \chi_\alpha G_m^{\ \alpha} \right) + b\, \omega^m \omega^n K_{mn},
\end{aligned} \tag{3.1}
$$

where $A_\alpha, \ldots, K_{mn}$ are arbitrary superfields of x_m, θ^α. It clearly makes a big difference that ω^m no longer is a 1-form. Notice that the requirement that $\mathcal{U}^{(1)}(z) \in \mathcal{H}_+$ forbids the terms $b\,(\lambda^\alpha \lambda^\beta F_{\alpha\beta} + \lambda^\alpha \xi^m F_{\alpha m})$ in the vertex (these terms are indeed present in [1, eq. (6.4)]).

The condition $\{Q, \mathcal{U}^{(1)}(z)\} = 0$ implies the following equations

$$
\begin{aligned}
D_{(\alpha} A_{\beta)} - \frac{1}{2}\gamma^m_{\alpha\beta} A_m &= 0, \\
\partial_m A_\alpha - D_\alpha A_m + \gamma_{m\alpha\beta} W^\beta &= 0, \\
\partial_{[m} A_{n]} + F_{mn} &= 0, \qquad D_\beta W^\alpha + F_\beta^{\ \alpha} = 0, \\
\partial_m W^\alpha + F_m^{\ \alpha} &= 0, \qquad F^{\alpha\beta} = 0, \\
D_\alpha B_m - G_{m\alpha} &= 0, \qquad \partial_n B_m - G_{mn} = 0, \\
G_m^{\ \alpha} &= 0, \qquad K_{mn} = 0,
\end{aligned} \tag{3.2}
$$

where $D_\alpha \equiv \partial/\partial\theta^\alpha + \frac{1}{2}\theta^\beta \gamma^m_{\alpha\beta}\partial/\partial x^m$.[2] The terms in $\{Q, \mathcal{U}^{(1)}(z)\}$ which contain the field b yield equations which are the Bianchi identities [1]. From the first two equations of (3.2)

[1] In [1] we presented four different solutions B^i of the the equation $T_{zz}(z) = \{Q, B^i_{zz}(z)\}$. None of the solutions B^i have definite grading except $B^{\rm IV}_{zz}(z) = b\,\hat{T}_{zz}(z) + b\partial_z b c_z - \frac{1}{2}\partial_z^2 b$ which has grading equal to -2 carried by the antighost b.

[2] The normalization is chosen such that $D_\alpha D_\beta + D_\beta D_\alpha = \gamma^m_{\alpha\beta}\partial_m$. We define $D_{(\alpha}A_{\beta)} = \frac{1}{2}(D_\alpha A_\beta + D_\beta A_\alpha)$ and $\partial_{[m}A_{n]} = \frac{1}{2}(\partial_m A_n - \partial_n A_m)$.

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one gets the field equations for $N=1, d=(9,1)$ SQED [8]

$$\gamma^{\alpha\beta}_{[mnpqr]} D_\alpha A_\beta = 0\,, \tag{3.3}$$

as well as the definition of the vector potential A_m and the spinorial field strength W^α in terms of A_α

$$A_m = \frac{1}{8}\gamma^{\alpha\beta}_m D_\alpha A_\beta\,, \qquad W^\alpha = \frac{1}{10}\gamma^{\alpha\beta}_m\left(D_\beta A_m - \partial_m A_\beta\right). \tag{3.4}$$

Moreover, the remaining equations in (3.2) imply that the curvatures F_{mn}, F^α_m, and F^α_β are expressed in terms of the spinor potential A_α, and similarly the superfields $G_{m\alpha}, G_{mn}$ are solely functions of B_m.

The gauge transformations of the vertex $\mathcal{U}^{(1)}(z)$ are generated by the BRST variation of spin zero field $\Omega^{(0)}(z) \in \mathcal{H}_+$ with ghost number zero, whose most general expression is given by $\Omega^{(0)}(z) = C + b\,\omega^m M_m$, with C and M_m arbitrary superfields. The BRST variation of $\Omega^{(0)}$ is

$$\delta\mathcal{U}^{(1)}(z) = \left[Q, \Omega^{(0)}(z)\right] = \lambda^\alpha D_\alpha C + \xi^m \partial_m C + \omega^m M_m + b\,\omega^m(\lambda^\alpha D_\alpha M_m + \xi^n \partial_n M_m)\,. \tag{3.5}$$

One can easily check that the field M_m can be used to gauge away B_m, while C is the usual parameter of the gauge transformations on the SQED potentials: $\delta A_\alpha = D_\alpha C, \delta A_m = \partial_m C$. Thus, the only independent superfield is A_α, and it satisfies (3.3) which is gauge invariant.

In order to exhibit explicitly the physical degrees of freedom it is convenient to work in the gauge $\theta^\alpha A_\alpha = 0$. The photon a_m and the photino u^α are identified with the first coefficients in the expansion of A_m and W^α in powers of θ

$$\begin{aligned} A_m(x,\theta) &= a_m(x) + \theta^\alpha \gamma_{m\alpha\beta} u^\beta(x) + \cdots\,, \\ W^\alpha(x,\theta) &= u^\alpha(x) + \gamma^{mn,\alpha}_{\beta}\theta^\beta \partial_m a_n(x) + \cdots\,. \end{aligned} \tag{3.6}$$

The ellipses denote terms which contain derivatives of a_m and u^α. From the Bianchi identities and (3.3) one derives $\partial_m F^{mn} = 0$ and $\gamma^m_{\alpha\beta}\partial_m W^\beta = 0$, while $\theta^\alpha A_\alpha = 0$ implies that $A_\alpha = \frac{1}{2}a^m(\gamma_m\theta)_\alpha + a^{m_1\dots m_5}(\gamma_{m_1\dots m_5}\theta)_\alpha + (\gamma^m\theta)_\alpha(u\gamma_m\theta) + (\gamma_{m_1\dots m_5}\theta)_\alpha v^{m_1\dots m_5}_\beta\theta^\beta + \cdots$. From (3.3) it follows that $a^{m_1\dots m_5} = v^{m_1\dots m_5}_\beta = 0$ and in this way one has obtained the linearized field equations for the gauge field a_m and the gaugino u^α.

At other ghost numbers the cohomology groups $H^{(n)}(Q,\mathcal{H}_+)$ vanish, except the group $H^{(2)}(Q,\mathcal{H}_+)$ which contains the antifields a^*_m, u^*_α of the gauge field a_m and of the gaugino u^α [7]. The analysis can be performed along the lines of the previous discussion and one can see that all the ω-dependent terms are cohomologically trivial, and therefore can be reabsorbed by a gauge transformation. For the ω-independent terms one has the following general decomposition

$$\begin{aligned} \mathcal{U}^{(2)} = {} & \lambda^\alpha\lambda^\beta A^*_{\alpha\beta} + \lambda^\alpha\xi^m A^*_{\alpha m} + \cdots + \chi_\alpha\chi_\beta A^{*,\alpha\beta} + \\ & + b\left(\lambda^\alpha\lambda^\beta\xi^m F^*_{\alpha\beta m} + \lambda^\alpha\xi^m\xi^n F^*_{\alpha mn} + \lambda^\alpha\lambda^\beta\chi_\gamma F^{*\ \gamma}_{\alpha\beta} + \cdots + \chi_\alpha\chi_\beta\chi_\gamma F^{*,\alpha\beta\gamma}\right). \end{aligned} \tag{3.7}$$

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The condition $\mathcal{U}^{(2)} \in \mathcal{H}_+$ does not allow the term $\lambda^\alpha\lambda^\beta\lambda^\gamma F^*_{\alpha\beta\gamma}$, which is permitted by ghost number counting. This implies that the superfield $A^*_{\alpha\beta}$ should satisfy the field equation $D_{(\alpha}A^*_{\beta\gamma)} - \frac{1}{2}\gamma^m_{(\alpha\beta}A^*_{\gamma)m} = 0$. Furthermore, the gauge transformations are generated by the BRST variation of a ghost-number 1 superfield $\Omega^{(1)}(z)$ (whose decomposition is given in eq. (3.1)), namely $\delta\mathcal{U}^{(2)}(z) \equiv \{Q, \Omega^{(1)}(z)\}$. One obtains that the gauge transformations of the super-antifield $A^*_{\alpha\beta}$ are the equations of motion of the potential A_α

$$\delta A^*_{\alpha\beta} = D_{(\alpha}A_{\beta)} - \frac{1}{2}\gamma^m_{\alpha\beta}A_m\,. \tag{3.8}$$

As shown in [2] the only states in the BRST cohomology at non-zero momentum are the on-shell gauge field a_m and the gluino u^α at ghost number $+1$ and the corresponding antifields a^*_m, u^*_α at ghost number $+2$. The latter are the coefficients of the superfield $A^*_{\alpha\beta}$. In fact, since $A^*_{\alpha\beta}$ is a symmetric bispinor, it can be decomposed into a vector part and a self-dual 5-form one: $A^*_{\alpha\beta} = \gamma^m_{\alpha\beta}A^*_m + \gamma^{[mnrpq]}_{\alpha\beta}A^*_{[mnrpq]}$. The gauge transformations (3.8) remove the vector component A^*_m, and the first coefficients of the θ-expansion of $A^*_{[mnrpq]}$ exhibit the on-shell fields

$$A^*_{[mnpqr]} = (\theta\gamma_{[mnp}\theta)(\theta\gamma_{qr]})^\alpha u^*_\alpha(x) + (\theta\gamma_{[mnp}\theta)(\theta\gamma_{qr]s})^\alpha a^{*,m}(x) + \cdots\,. \tag{3.9}$$

The ellipses involve terms with derivatives of a^*_m and u^*_α.

Before closing this section, we point out that the choice of the subspace $\mathcal{H}_+$ is one of the possible choices. Another interesting situation is the restriction to the subspace $\mathcal{H}'_+$ with strictly positive grading. It is straightforward to see that, at ghost number 1, the cohomology $H^{(1)}(Q, \mathcal{H}'_+)$ is identified with the gauge potentials which satisfy $F_{mn} = D_\alpha W^\beta = 0$. This corresponds to a topological gauge model in the target space.

4. The massless spectrum of the closed superstring

According to our formalism [1], the BRST charge is the sum of the BRST charge for the left and right movers denoted by Q_L and Q_R, respectively. In the super-Poincaré covariant formulation of closed superstring, one can choose the spinors θ^α_L and $\theta^{\hat\alpha}_R$ to be Weyl or anti-Weyl in the left- or in the right-mover sector: the same chirality for both spinors leads to type-IIB strings, opposite chiralities lead to type-IIA strings. In the following we will not distinguish between the two cases.

As in [2], also in our formalism the holomorphic and anti-holomorphic sector are decoupled. Each operator in the left-mover sector has its counterpart in the right-mover one. Therefore, the assignments of ghost number and grading, as well as the central charge cancellations (cf. eq. (2.4)), are exactly the same in both sectors.

Denoting by $\mathcal{H}^L_+$ and $\mathcal{H}^R_+$ the left- and right-mover subspaces with non-negative gradings, the physical state condition is expressed by

$$\begin{aligned} Q_L|\psi\rangle &= Q_R|\psi\rangle = 0\,, \qquad |\psi\rangle \in \mathcal{H}^L_+ \otimes \mathcal{H}^R_+\,, \\ |\psi\rangle &\neq Q_R|\phi\rangle + Q_L\,|\chi\rangle\,, \qquad |\phi\rangle, |\chi\rangle \in \mathcal{H}^L_+ \otimes \mathcal{H}^R_+\,, \\ Q_R\,|\chi\rangle &= Q_L|\phi\rangle = 0\,. \end{aligned} \tag{4.1}$$

The cohomology group $H^{(1,1)}(Q, \mathcal{H}_+^L \otimes \mathcal{H}_+^R)$ is identified with the physical degrees of freedom. In particular, the spin zero vertex operator $\mathcal{U}^{(1,1)}$ contains the massless fields of the closed superstring. To determine the linearized field equations, we first analyze the $\omega, \hat{\omega}$-dependent terms of the vertex. Since ω and $\hat{\omega}$ are BRST inert, one can analyze the sectors of $H^{(1,1)}(Q, \mathcal{H}_+^L \otimes \mathcal{H}_+^R)$ with different powers of ω and $\hat{\omega}$ separately. We prove that all the sectors with positive powers of ω and $\hat{\omega}$ vanish. Consider the generic decomposition with one power of ω, $\mathcal{U}^{(1,1)} = \omega^m(\mathcal{B}_m(\hat{C}) + b\, C^A \mathcal{F}_{mA}(\hat{C}))$, where C^A and $\hat{C}^{\hat{A}}$ denote collectively the ghost fields $(\lambda^\alpha, \xi^m, \chi_\alpha, b)$ of left- and right-moving sectors. The functions $\mathcal{B}_m$ and $\mathcal{F}_{mA}$ are polynomials in the right-moving ghost fields $\hat{C}^{\hat{A}}$ with superfield coefficients.

The conditions $[Q_L, \mathcal{U}^{(1,1)}] = [Q_R, \mathcal{U}^{(1,1)}] = 0$ imply that

$$Q_L\, \mathcal{B}_m(\hat{C}) - C^A \mathcal{F}_{mA}(\hat{C}) = 0\,, \qquad Q_R\, \mathcal{B}_m(\hat{C}) = Q_R\, \mathcal{F}_{mA}(\hat{C}) = 0\,. \tag{4.2}$$

Thus, the vertex $\mathcal{U}^{(1,1)} = \{Q_L, \Lambda\}$ is BRST trivial where $\Lambda = (b\, \omega^m \mathcal{B}_m(\hat{C}))$ and $\{Q_R, \Lambda\} = 0$. The same argument can be easily repeated for any positive power of ω or $\hat{\omega}$. Therefore, we have to analyze only the $\omega, \hat{\omega}$-independent terms.

The vertex $\mathcal{U}$ can be decomposed into the following terms

$$\begin{aligned}
\mathcal{U}(z,\bar{z}) &= \mathcal{U}^{(1,1)} + b\,\mathcal{U}^{(2,1)} + \hat{b}\,\mathcal{U}^{(1,2)} + b\,\hat{b}\,\mathcal{U}^{(2,2)}\,, \\
\mathcal{U}^{(1,1)} &= \left(\lambda_L^\alpha \lambda_R^{\hat{\alpha}} A_{\alpha\hat{\alpha}} + \cdots + \chi_{L\alpha}\chi_{R\hat{\alpha}} A^{\alpha\hat{\alpha}}\right), \\
\mathcal{U}^{(2,1)} &= \left(\xi_L^m \xi_L^n \lambda_R^{\hat{\alpha}} B_{mn\hat{\alpha}} + \cdots + \chi_{L\alpha}\chi_{L\beta}\chi_{R\hat{\alpha}} B^{\alpha\beta\hat{\alpha}}\right), \\
\mathcal{U}^{(1,2)} &= \left(\lambda_L^\alpha \xi_R^{\hat{m}} \xi_R^{\hat{n}} B_{\alpha\hat{m}\hat{n}} + \cdots + \chi_\alpha \chi_{R\hat{\alpha}}\chi_{R\hat{\gamma}} B^{\alpha\hat{\alpha}\hat{\gamma}}\right), \\
\mathcal{U}^{(2,2)} &= \left(\xi_L^m \xi_L^n \xi_R^{\hat{m}} \xi_R^{\hat{n}} C_{mn\hat{m}\hat{n}} + \cdots + \chi_{L\alpha}\chi_{L\beta}\chi_{R\hat{\gamma}}\chi_{R\hat{\epsilon}} C^{\alpha\beta\hat{\gamma}\hat{\epsilon}}\right),
\end{aligned} \tag{4.3}$$

where all the coefficients $A_{\alpha\hat{\alpha}}, \dots C^{\alpha\beta\hat{\gamma}\hat{\epsilon}}$ are arbitrary superfields. Notice that the condition $\mathcal{U}(z,\bar{z}) \in \mathcal{H}_+^L \otimes \mathcal{H}_+^R$ forbids the terms $b\, \lambda_L^\alpha \lambda_L^\beta, b\, \lambda_L^\alpha \xi_L^m$ and the corresponding ones for right-movers. Finally, also the terms $b\, \hat{b}\, \lambda_L^\alpha \lambda_L^\beta \lambda_R^{\hat{\alpha}} \lambda_R^{\hat{\gamma}}, \dots, b\, \hat{b}\, \lambda_L^\alpha \xi_L^m \lambda_R^{\hat{\alpha}} \xi_R^{\hat{m}}$ are ruled out.

As in the case of the open superstring, the absence of these terms combined with the BRST invariance yields the following equations[3]

$$\begin{aligned}
D_{(\alpha} A_{\beta)\hat{\alpha}} - \frac{1}{2}\gamma^m_{\alpha\beta} A_{m\hat{\alpha}} &= 0\,, & D_{(\hat{\alpha}} A_{\alpha\hat{\gamma})} - \frac{1}{2}\gamma^m_{\hat{\alpha}\hat{\gamma}} A_{\alpha m} &= 0\,, \\
\partial_m A_{\alpha\hat{\alpha}} - D_\alpha A_{m\hat{\alpha}} + \gamma_{m\alpha\beta} A^\beta_{\ \hat{\alpha}} &= 0\,, & \partial_m A_{\alpha\hat{\alpha}} - D_{\hat{\alpha}} A_{\alpha m} + \gamma_{m\hat{\alpha}\hat{\gamma}} A_\alpha^{\ \hat{\gamma}} &= 0\,, \\
D_{(\alpha} A_{\beta)m} - \frac{1}{2}\gamma^n_{\alpha\beta} A_{mn} &= 0\,, & D_{(\hat{\alpha}} A_{m\hat{\gamma})} - \frac{1}{2}\gamma^n_{\hat{\alpha}\hat{\gamma}} A_{mn} &= 0\,, \\
\partial_m A_{\alpha n} - D_\alpha A_{mn} + \gamma_{m\alpha\beta} A^\beta_{\ n} &= 0\,, & \partial_m A_{n\hat{\alpha}} - D_{\hat{\alpha}} A_{mn} + \gamma_{m\hat{\alpha}\hat{\gamma}} A_n^{\ \hat{\gamma}} &= 0\,, \\
D_{(\alpha} A^{\hat{\gamma}}_{\ \beta)} - \frac{1}{2}\gamma^m_{\alpha\beta} A^{\hat{\gamma}}_{\ m} &= 0\,, & D_{(\hat{\alpha}} A^\alpha_{\ \hat{\gamma})} - \frac{1}{2}\gamma^m_{\hat{\alpha}\hat{\gamma}} A^\alpha_{\ m} &= 0\,, \\
\partial_m A_\alpha^{\ \hat{\gamma}} - D_\alpha A_m^{\ \hat{\gamma}} + \gamma_{m\alpha\beta} A^{\beta\hat{\gamma}} &= 0\,, & \partial_m A^\alpha_{\ \hat{\alpha}} - D_{\hat{\alpha}} A^\alpha_{\ m} + \gamma_{m\hat{\alpha}\hat{\gamma}} A^{\alpha\hat{\gamma}} &= 0\,.
\end{aligned} \tag{4.4}$$

The other equations coming from $[Q_L + Q_R, \mathcal{U}] = 0$ will not impose any further constraint on the superfields, but they define consistently the curvature in terms of $A_{\alpha\hat{\alpha}}, \dots A^{\alpha\hat{\alpha}}$.

[3]The N=2 D=9,1 supersymmetric derivatives are defined by $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta\partial_m$ and $\hat{D}_{\hat{\alpha}} = \frac{\partial}{\partial\hat{\theta}^{\hat{\alpha}}} + \frac{1}{2}\gamma^m_{\hat{\alpha}\hat{\beta}}\hat{\theta}^{\hat{\beta}}\partial_m$.

Notice that the superfields $A_{m\hat{\alpha}}, A_{\alpha m}, \ldots A^{\alpha\hat{\alpha}}$ are solved in terms of the fundamental field $A_{\alpha\hat{\alpha}}$ as follows

$$
\begin{aligned}
A_{m\hat{\alpha}} &= \frac{1}{8} D_\alpha \gamma_m^{\alpha\beta} A_{\beta\hat{\alpha}}\,, \qquad A_{\alpha m} = \frac{1}{8} D_{\hat{\alpha}} \gamma_m^{\hat{\alpha}\hat{\beta}} A_{\alpha\hat{\beta}}\,, \\
A^{\alpha}_{\hat{\alpha}} &= \frac{1}{10}\gamma^{m\alpha\beta}\left(D_\beta A_{m\hat{\alpha}} - \partial_m A_{\beta\hat{\alpha}}\right), \qquad A^{\hat{\alpha}}_{\alpha} = \frac{1}{10}\gamma^{m\hat{\alpha}\hat{\gamma}}\left(D_{\hat{\gamma}} A_{\alpha m} - \partial_m A_{\alpha\hat{\gamma}}\right), \\
A^{\hat{\gamma}}_{m} &= \frac{1}{8}\gamma_m^{\alpha\beta} D_\alpha A^{\hat{\gamma}}_{\beta}\,, \qquad A^{\alpha}_{m} = \frac{1}{8}\gamma_m^{\hat{\alpha}\hat{\gamma}} D_{\hat{\alpha}} A^{\alpha}_{\hat{\gamma}}\,, \\
A_{mn} &= \frac{1}{64} D_\alpha D_{\hat{\alpha}} \gamma_m^{\alpha\beta}\gamma_n^{\hat{\alpha}\hat{\beta}} A_{\beta\hat{\beta}}\,, \qquad A^{\alpha\hat{\alpha}} = \frac{1}{10}\gamma^{m\alpha\beta}\left(D_\beta A^{\hat{\alpha}}_{m} - \partial_m A^{\hat{\alpha}}_{\beta}\right).
\end{aligned} \tag{4.5}
$$

The superfield $A_{\alpha\hat{\alpha}}$ satisfies the field equations

$$
\gamma^{\alpha\beta}_{mnpqr} D_\alpha A_{\beta\hat{\gamma}} = 0\,, \qquad \gamma^{\hat{\alpha}\hat{\gamma}}_{mnpqr} D_{\hat{\alpha}} A_{\gamma\hat{\gamma}} = 0\,, \tag{4.6}
$$

which are the linearized $N = 2$ supergravity equations of motion. The gauge transformations are generated by the BRST variations of two generic superfields $\Omega^{(0,1)}$ and $\Omega^{(1,0)}$ by $\delta\mathcal{U}(z,\bar{z}) = \{Q_L, \Omega^{(0,1)}\} + \{Q_R, \Omega^{(1,0)}\}$, with $\{Q_R, \Omega^{(0,1)}\} = \{Q_L, \Omega^{(1,0)}\} = 0$ and $\Omega^{(0,1)} \in \mathcal{H}_+$, $\Omega^{(1,0)} \in \mathcal{H}_+$. Assuming for $\Omega^{(0,1)}$ the general decomposition

$$
\begin{aligned}
\Omega^{(0,1)}(z) &= \lambda^{\hat{\alpha}}_R \hat{\Lambda}_{\hat{\alpha}} + \xi^m_R \hat{\Lambda}_m + \chi_{R\hat{\alpha}} \hat{\Lambda}^{\hat{\alpha}} + \\
&\quad + b_R \left(\xi^m_R \xi^n_R \hat{\Xi}_{mn} + \lambda^{\hat{\alpha}}_R \chi_{R\hat{\gamma}} \hat{\Xi}^{\hat{\gamma}}_{\hat{\alpha}} + \chi_{R\hat{\alpha}}\, \xi^m \hat{\Xi}^{\hat{\alpha}}_m + \chi_{R\hat{\alpha}} \chi_{R\hat{\gamma}} \hat{\Xi}^{\hat{\alpha}\hat{\gamma}}\right),
\end{aligned} \tag{4.7}
$$

and analogously for $\Omega^{(1,0)}$ exchanging $R \to L$ and $\hat{\Lambda}_{\hat{\alpha}}, \ldots, \hat{\Xi}^{\hat{\alpha}\hat{\gamma}} \to \Lambda_\alpha, \ldots, \Xi^{\alpha\gamma}$, the relevant gauge transformations are

$$
\delta A_{\alpha\hat{\alpha}} = D_\alpha \hat{\Lambda}_{\hat{\alpha}} + D_{\hat{\alpha}} \Lambda_\alpha\,. \tag{4.8}
$$

The conditions $\{Q_R, \Omega^{(0,1)}\} = \{Q_L, \Omega^{(1,0)}\} = 0$ imply that the gauge parameters Λ_β and $\hat{\Lambda}_{\hat{\gamma}}$ satisfy the following equations

$$
\gamma^{\alpha\beta}_{[mnpqr]} D_\alpha \Lambda_\beta = 0\,, \qquad \gamma^{\hat{\alpha}\hat{\gamma}}_{[mnpqr]} D_{\hat{\alpha}} \hat{\Lambda}_{\hat{\gamma}} = 0\,. \tag{4.9}
$$

The gauge transformations for the other superfields $A_{m\hat{\alpha}}, \ldots, A^{\alpha\hat{\alpha}}$ can be easily derived from eq. (4.9) using their definitions (4.5). The field eqs. (4.6) and the gauge transformations (4.8) derived for the closed superstring are in agreement with [2, 6]. The physical degrees of freedom can be easily read from the first components of the θ^α_L and $\theta^{\hat{\alpha}}_R$ of the superfields $A_{mn}(x, \theta_L, \theta_R)$, $A^{\hat{\gamma}}_m(x, \theta_L, \theta_R)$, $A^\alpha_m(x, \theta_L, \theta_R)$ and $A^{\alpha\hat{\alpha}}(x, \theta_L, \theta_R)$. The first component of $A_{mn} = (g_{mn} + b_{mn} + \eta_{mn}\phi) + \mathcal{O}(\theta_L, \theta_R)$ describe the graviton, the NS-NS two-form and the dilaton. The first components of $A^{\hat{\gamma}}_m = \psi^{\hat{\gamma}}_m + \mathcal{O}(\theta_L, \theta_R)$ and $A^\alpha_m = \psi^\alpha_m + \mathcal{O}(\theta_L, \theta_R)$ contain the $N = 2$ gravitinos $\psi^{\hat{\gamma}}_m, \psi^\alpha_m$, and the chirality of the right-mover spinor θ_R determines if the string is type IIA or IIB. Finally, $A^{\alpha\hat{\alpha}} = F^{\alpha\hat{\alpha}} + \mathcal{O}(\theta_L, \theta_R)$, where $F^{\alpha\hat{\alpha}}$ are the R-R field strenghts. The linearized field equations for $g_{mn}, b_{mn}, \phi, \psi^\alpha_m, \psi^{\hat{\gamma}}_m$ and $F^{\alpha\hat{\alpha}}$ are discussed in great detail in [6].

Of course the closed superstring massless spectrum can be understood as the tensor product of the cohomologies for open superstring of left- and right-movers. Mathematically, this is encoded in the well-known Künneth formula $H^{(1,1)}(Q_L + Q_R, \mathcal{H}^L_+ \otimes \mathcal{H}^R_+) = H^{(1)}(Q_L, \mathcal{H}^L_+) \otimes H^{(1)}(Q_R, \mathcal{H}^R_+)$.

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For applications to correlation functions we expect to need not only local vertex operators such as $\mathcal{U}^{(1)}$ for the open string and $\mathcal{U}^{(1,1)}$ for the closed strings, but also integrated vertex operators $\int dz \mathcal{V}_z^{(0,1)}, \int d\bar{z} \mathcal{V}_{\bar{z}}^{(1,0)}$, and $\int dz d\bar{z} \mathcal{V}_{z\bar{z}}^{(0,0)}$. These vertex operators satisfy descent equations. Introducing the notation $\mathcal{O}_{c,d}^{(a,b)}$ for local vertex operators with ghost number $a(b)$ in the left (right) sector and (anti) holomorphic indices $c(d)$, we identify

$$\mathcal{O}_{0,0}^{(1,1)} = \mathcal{U}^{(1,1)}\,, \qquad \mathcal{O}_{1,0}^{(0,1)} = \mathcal{V}_z^{(0,1)}\,, \qquad \mathcal{O}_{0,1}^{(1,0)} = \mathcal{V}_{\bar{z}}^{(1,0)}\,, \qquad \mathcal{O}_{1,1}^{(0,0)} = \mathcal{V}_{z\bar{z}}^{(0,0)}\,. \tag{4.10}$$

The descent equations read

$$\left[Q_L, \mathcal{O}_{c,d}^{(a,b)}\right] = d\,\mathcal{O}_{c-1,d}^{(a+1,b)}\,, \qquad \left[Q_R, \mathcal{O}_{c,d}^{(a,b)}\right] = \bar{d}\,\mathcal{O}_{c,d-1}^{(a,b+1)}\,, \tag{4.11}$$

where d and $\bar{d}$ are the holomorphic and antiholomorphic differentials. More explicitly, at the first level we have

$$\left[Q_L, \mathcal{U}^{(1,1)}\right] = 0\,, \qquad \left[Q_R, \mathcal{U}^{(1,1)}\right] = 0\,, \tag{4.12}$$

while at the next level, we get

$$\begin{aligned} \left[Q_L, \mathcal{V}_z^{(0,1)}\right] &= \partial_z \mathcal{U}^{(1,1)}\,, \qquad \left[Q_R, \mathcal{V}_z^{(0,1)}\right] = 0\,, \\ \left[Q_R, \mathcal{V}_{\bar{z}}^{(1,0)}\right] &= \partial_{\bar{z}} \mathcal{U}^{(1,1)}\,, \qquad \left[Q_L, \mathcal{V}_{\bar{z}}^{(1,0)}\right] = 0\,, \end{aligned} \tag{4.13}$$

and, finally,

$$\left[Q_L, \mathcal{V}_{z\bar{z}}^{(0,0)}\right] = \partial_z \mathcal{V}_{\bar{z}}^{(1,0)}\,, \qquad \left[Q_R, \mathcal{V}_{z\bar{z}}^{(0,0)}\right] = \partial_{\bar{z}} \mathcal{V}_z^{(0,1)}\,. \tag{4.14}$$

The unintegrated vertex operator $\mathcal{U}^{(1,1)}$ was given in (4.3). One can write down similar expressions for the once or twice integrated vertex operators. To simplify the expressions we introduce a matrix notation for each of the four terms in (4.3). For $\mathcal{U}^{(1,1)}$ we write $\mathcal{U}^{(1,1)} = E^M E^{\hat{N}} \mathcal{M}_{M\hat{N}}(x, \theta_L, \theta_R)$, where $E^M = (\lambda^\alpha, \xi^m, \chi_\alpha)$, $E^{\hat{N}} = (\lambda^{\hat{\beta}}, \hat{\xi}^n, \chi_{\hat{\beta}})$ and $\mathcal{M}_{M\hat{N}}(x, \theta_L, \theta_R)$ is given by

$$\mathcal{M}_{M\hat{N}}(x, \theta_L, \theta_R) = \begin{pmatrix} A_{\alpha\hat{\beta}} & A_{\alpha n} & A_\alpha^{\hat{\beta}} \\ A_{m\hat{\beta}} & A_{mn} & A_m^{\hat{\beta}} \\ A^\alpha_{\hat{\beta}} & A^\alpha_n & A^{\alpha\hat{\beta}} \end{pmatrix}. \tag{4.15}$$

Similar expressions can be written down for $\mathcal{U}^{(2,1)}, \mathcal{U}^{(1,2)}$ and $\mathcal{U}^{(2,2)}$, but we shall not need them. Furthermore we do not introduce any term with either ω^m and $\bar{\omega}^m$ for reasons explained before.

For $\mathcal{V}_z^{(0,1)}$, we have to introduce two further ingredients: a vector of the form $F_{z,M} = (\partial_z \theta^\alpha, \Pi_{zm}, d_{z\alpha})$ and a tensor of the form $T_{z,MN} = (w_{z\alpha}\lambda^\beta, w_{z\alpha}\xi^m, w_{z\alpha}\chi_\beta, \beta_{zm}\xi^n,\ \beta_{z\mu}\chi_\beta,\ \kappa_z^\alpha \chi_\beta)$. The vertex $\mathcal{V}_z^{(0,1)}$ is then given by

$$\mathcal{V}_z^{(0,1)} = F_z^M E^{\hat{N}}\, \mathcal{M}'_{M\hat{N}}(x, \theta_L, \theta_R) + T_z^{MN}\, E^{\hat{P}}\, \mathcal{N}_{MN,\hat{P}}(x, \theta_L, \theta_R)\,. \tag{4.16}$$

Note that no term $c_z b$ is included in $T_{z,MN}$ because it belongs to the sector $b\mathcal{V}_z^{(1,1)}$. Finally, $\mathcal{V}_{z\bar{z}}^{(0,0)}$ is represented by

$$\mathcal{V}_{z\bar{z}}^{(0,0)} = F_z^{\ M}\, F_{\bar{z}}^{\ \hat{N}}\, \mathcal{M}''_{M\hat{N}}(x,\theta_L,\theta_R) + T_z^{\ MN}\, F_{\bar{z}}^{\ \hat{P}}\, \mathcal{N}_{MN,\hat{P}}(x,\theta_L,\theta_R) + \\ + F_z^{\ M}\, T_{\bar{z}}^{\ \hat{P}\hat{Q}}\, \bar{\mathcal{N}}_{M,\hat{P}\hat{Q}}(x,\theta_L,\theta_R) + T_z^{\ MN}\, T_{\bar{z}}^{\ \hat{P}\hat{Q}}\, \mathcal{P}_{MN,\hat{P}\hat{Q}}(x,\theta_L,\theta_R)\,. \tag{4.17}$$

Uisng the descent equations, one can derive the remarkable simple result

$$\mathcal{M}_{M\hat{N}}(x,\theta_L,\theta_R) = \mathcal{M}'_{M\hat{N}}(x,\theta_L,\theta_R) = \mathcal{M}''_{M\hat{N}}(x,\theta_L,\theta_R) \tag{4.18}$$

which allows us to reconstruct completely the other pieces of the vertex, namely the superfields $\mathcal{N}_{MN,\hat{P}}(x,\theta_L,\theta_R)$, $\bar{\mathcal{N}}_{M,\hat{P}\hat{Q}}(x,\theta_L,\theta_R)$ and $\mathcal{P}_{MN,\hat{P}\hat{Q}}(x,\theta_L,\theta_R)$ in terms of the superfields contained in the matrix $\mathcal{M}_{M\hat{N}}(x,\theta_L,\theta_R)$.

5. Conclusions

We conclude by giving an argument for the presence of the term $-\eta_z^m\bar{\partial}\omega_m$ in the action, with (η_z^m,ω_m) a ghost-charge $(-1,1)$ spin $(1,0)$ system. We start from the observation that in the approach with the pure spinor constraint $\lambda^\alpha\gamma^m_{\alpha\beta}\lambda^\beta = 0$ the action with $\mathcal{L} = -\beta_{z,\alpha}\bar{\partial}\lambda^\alpha$ has the local gauge invariance

$$\delta\beta_{z,\alpha} = \Lambda_{zm}\gamma^m_{\alpha\beta}\lambda^\beta\,, \qquad \delta\lambda^\alpha = 0\,. \tag{5.1}$$

The gauge parameter has ghost number -2. To remove the pure spinor constraint we introduce the Lagrange multiplier field $\alpha_{z\bar{z},m}$ with the ghost number -2

$$\mathcal{L} = -\beta_{z,\alpha}\bar{\partial}\lambda^\alpha + \frac{1}{2}\alpha_{z\bar{z},m}\lambda^\alpha\gamma^m_{\alpha\beta}\lambda^\beta\,. \tag{5.2}$$

The action with unconstrained λ^α is still gauge invariant if $\alpha_{z\bar{z},m}$ transforms as $\delta\alpha_{z\bar{z},m} = -\bar{\partial}\Lambda_{zm}$. We fix this local gauge symmetry as usual by adding a BRST exact term. The gauge parameter Λ_{zm} becomes a field η_{zm} with ghost number -1, which is an antighost. As the BRST exact term we take

$$s\,(\omega^m\alpha_{z\bar{z},m}) = d^m\alpha_{z\bar{z},m} - \omega^m\bar{\partial}\eta_{zm}\,. \tag{5.3}$$

The field ω^m has ghost number $+1$, and is a ghost field; it transforms into a BRST auxiliary field d^m with ghost number $+2$. Integrating over d^m and $\alpha_{z\bar{z},m}$ sets these fields to zero, and one is indeed left with the term $\bar{\partial}\eta_{zm}\omega^m = -\eta_{zm}\bar{\partial}\,\omega^m$ in the action. We end up by recalling that the previous argument is a strong indication in favour of the presence of the fields η_z^m and ω_m in our formalism, nevertheless a complete picture can be only achieved by starting from a gauge invariant action and using the Batalin-Vilkovisky framework for its quantization.

Before concluding, we would like to point out the relation between the cohomology in the Berkovits' approach and in our formalism which also indicates the path towards the proof of the equivalence of the two approaches.

The group $H(Q_B|\mathcal{H}_{p.s.})$, which represents the cohomology in the Berkovits' formalism is an example of a constrained BRST cohomology, or equivalently, of equivariant cohomology [9]. In the latter case, the BRST cohomology is computed on the supermanifold M with coordinates x^m,θ^α on which the space-time translations $x^m \to x^m + \frac{1}{2}\lambda\gamma^m\lambda$,

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generated by unconstrained spinors λ^α, act freely. One finds that $Q_B^2 = -\mathcal{L}_V$, where $\mathcal{L}_V$ is the Lie dierivative $\mathcal{L}_V = d\,\iota_V + \iota_V\, d$ and ι_V is the contraction of a form with the vector $V^m = \frac{1}{2}\lambda\gamma^m\lambda$.

One can represent ι_V by the operator $\oint dz\, V^m \beta_{zm}$; its action of (parity reversed forms) ξ^m is then given by the OPE of $\beta_{zm}(z)$ with $\xi^m(z)$. The exterior differential d is $\xi^m\partial_m$, where ξ^m are the parity-reversed coordinates of the cotagent bundle $\Pi T^*\mathcal{M}$. The usual exterior derivative $d = dx^m\partial_m$ has been replaced by $-\oint dz\, \xi^m \Pi_{zm}$. Since $\Pi_z^m(z)\partial^l x^n(w) \sim (z-w)^{-l-1}$, the operator $-\oint dz\, \xi^m\Pi_{zm}$ represents the exterior derivative on the jet bundle $\{x^m, \partial x^m, \partial^2 x^m, \ldots\}$. Following the approach of equivariant cohomology [9], one can define a new BRST operator Q' by

$$Q' = Q_B + d + \iota_V = Q - \oint \xi^m \Pi_{zm} - \oint \frac{1}{2}\lambda^\alpha \gamma^m_{\alpha\beta}\lambda^\beta \beta_{zm}\,. \tag{5.4}$$

Unfortunately, this operator fails to be nilpotent and the solution of this problem has been discussed in [1]. Moreover, it turns out that the BRST cohomology computed in the space of vertex operators with vanishing conformal weight and non-negative gradings coincides with the massless spectrum of the superstring, and this suggests a complete equivalence of the two approaches. It has been also pointed out in similar examples discussed in [10] that one needs further conditions on the functional space to identify the correct physical observables as elements of the BRST cohomology.

In the present paper, we have identified the massless spectrum for the open and closed superstrings by the cohomology at ghost number +1 of the BRST operator restricted to a subspace of the linear space of vertex operators. The subspace is selected by means of a natural grading which is assigned to the ghosts and antighosts.

Note added: In this article we have introduced a new quantum number for fields, the grading, and a new definition of physical states which required that vertex operators have non-negative grading. The results (the correct physical spectrum) justify to some extend this notion of a grading, and we also noted that a similar notion of grading is already needed in the RNS spinning string to obtain a non-trivial spectrum [11]. However, the referee has asked us to provide "a more extensive justification of the grading condition". Although this has not even been achieved for the RNS string, we have tried to come up with a deeper explanation of the grading, and now present a new understanding: *the grading number is the pure ghost number (resolution degree) of homological perturbation theory.*

According to homological perturbation theory [12], once one has an initial BRST-like symmetry s_0 which is nilpotent modulo constraints G_a and gauge transformations (with possibly field dependent parameter ϵ^a, where the gauge transformations are due to OPE's of the fields with the constraints), we may introduce new fields $\mathcal{P}_a$ and a new nilpotent operator[4] δ_{-1} such that

$$\delta_{-1}\mathcal{P}_a = -G_a\,, \qquad \delta_{-1}(\text{other fields}) = 0\,. \tag{5.5}$$

The new fields $\mathcal{P}_a$ carry a new quantum number usually called antifield number and the operator δ_{-1} lowers this number. It has been proven that one can always add further

[4]This operator is known in the literature as the Koszul-Tate resolutor.

operators $s_1, s_2, \ldots$ to $\delta_{-1} + s_0$ such that $s \equiv \delta_{-1} + s_0 + s_1 + s_2 + \cdots$ is nilpotent. The form of s_n with $n > 1$ follows from the requirement that $s^2 = 0$. The s_n's have definite antifield number equal to n. In addition, one can define a further quantum number by a linear combination of the antifield number and the ghost number; it turns out that the *pure ghost number* n_{pg}, defined as the sum of the ghost number n_g plus the antifield number n_{af}, coincides with our grading number as we explain in the following.

In the superstring case, s_0 should be identified with the Berkovits' BRST-like symmetry which acts on the fields $\Phi = (\theta^\alpha, x^m, d_\alpha, \lambda^\alpha, \beta_\alpha)$, and s_0 is nilpotent up to the pure spinor constraint $\lambda\gamma^m\lambda = 0$ and up to the gauge trasformations $\Delta_\epsilon \Phi(w) = \oint dz (\epsilon_m \lambda\gamma^m\lambda)(z)\Phi(w)$ (so the bracket $[\Phi, G_a]$ is in our case written in terms of the operator product). Indeed,

$$\begin{aligned} &Q_B^2 x^m = -\frac{1}{2}\lambda\gamma^m\lambda\,, \qquad Q_B^2\theta^\alpha = 0\,, \qquad Q_B^2 d_\alpha = -\frac{1}{2}\partial(\lambda\gamma^m\lambda)\,, \\ &Q_B^2\lambda^\alpha = 0\,, \qquad Q_B^2 w_\alpha = \Pi_m(\gamma^m\lambda)_\alpha = \Delta_\Pi w_\alpha\,. \end{aligned} \tag{5.6}$$

The field $\mathcal{P}_a$ corresponds in our case to ξ_m and $\delta_{-1}\mathcal{P}_a = -G_a$ corresponds to $\{Q_0, \xi^m\} = -\frac{1}{2}\lambda\gamma^m\lambda$, where $Q_0 = \oint dz j_z^{B,(0)}(z)$ given in (2.6). Further, we identify $Q_2 \equiv s_1$, $Q_3 = s_2$ and $Q_4 = s_3$, where Q_2, Q_3 and Q_4 are the charges corresponding to the currents $j_z^{B,(1)}(z), j_z^{B,(3/2)}(z)$ and $j_z^{B,(2)}(z)$, respectively.

The fields Φ have by definition vanishing antifield number. Hence $n_{pg}(\lambda) = n_g(\lambda) + n_{af}(\lambda) = 1 + 0 = 1$ which agrees (up to a factor 2) with our grading. Similarly also for all other fields n_{pg} is equal to twice our grading.

There is however, a difference between our approach and standard homological perturbation theory. In the latter case one has by definition only fields with positive antifield number (contributing to s_n with $n \geq 0$), but in our case we have antighosts in the theory, and if the ghosts have positive antifield number, it is reasonable to assign the opposite (negative) antifield number to the antighosts (in this way the action is neutral). One must introduce a floor from which to work upwards, in a similar way as Dirac introduced the concept of a sea to excluded unbounded negative energy. We have constructed such a floor by hand, by requiring that the vertex operators have a lower bound on their grading; from the previous correspondence it even follows that this lower bound is zero. This is at the present time the best we can do to explain why one should require that the vertex operators have non-negative grading.

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